Brownian Motion
Part II - Limiting Distribution Of A Scaled Symmetric Random Walk

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In Part II of The Brownian Motion series we will develop the mathematics that proves that as the step size in a scaled symmetric random walk goes to zero the distribution of the value of the random walk at time t converges to a normal distribution with mean zero and variance t where t is the length of the time interval over which both the random walk and the Brownian motion travel. If the moment generating function of the limiting distribution of a scaled symmetric random walk equals the moment generating function of a normal distribution then we can conclude that the probability distributions are indeed the same. If the probability distributions are the same then as step size goes to zero (and accordingly the number of steps within the specified time interval goes to infinity) then a scaled symmetric random walk, which is a discrete time process, converges to a Brownian motion, which is a continuous time process.

The Scaled Symmetric Random Walk

Imagine that we have a time interval [0, t] where $t > 0$. This time interval consists of t individual time periods where each time period is divided into n equally-sized subintervals. An example might be the time interval [0, 2.5] where $t = 2.5$ years and each year is divided into $n = 12$ months. Each time step within the time interval [0, t] is therefore $1 \div n$ in length. A random walk over this time interval would consist of $nt = 30$ time steps. If at each time step the random walk can either increase by one with probability 0.50 or decrease by one with probability 0.50 then the random walk becomes a symmetric random walk with mean zero and variance one. If each increment of the symmetric random walk is multiplied by a scalar constant then the symmetric random walk becomes a scaled symmetric random walk with mean zero and variance equal to the square of the scalar constant.

We will define k to be the end of any given time step in the scaled symmetric random walk such that k can be any integer value between one and nt. The values that k can take in set notation are...

$$k = \{1, 2, 3, ..., (n-1)t, nt\} \tag{1}$$

We will define X_k to be a random variable that can take the value of positive one (with probability 0.50) or negative one (with probability 0.50) at the end of any given time step k. We will define the scalar constant for our random walk to be the square root of time step length. If M_0 is the value of the random walk at time zero and M_k is the value of the random walk at the end of any given subinterval k then the equation for the value of the scaled symmetric random walk at the end of the time interval [0, t] is...

$$M_t = M_0 + \sum_{k=1}^{nt} \frac{1}{\sqrt{n}} X_k = M_0 + \frac{1}{\sqrt{n}} \sum_{k=1}^{nt} X_k \tag{2}$$

If we define the value of the random walk at time zero to be equal to zero then Equation (2) can be rewritten as...

$$M_t = \frac{1}{\sqrt{n}} \sum_{k=1}^{nt} X_k \tag{3}$$

The Moment Generating Function Of The Scaled Symmetric Random Walk

The moment generating function of the scaled symmetric random walk as defined by Equation (3) is...

$$\varphi(z) = \mathbb{E}\Big[e^{\,zM_t}\Big] \tag{4}$$

After substituting Equation (3) into Equation (4) we can rewrite the moment generating function for the scaled symmetric random walk as...

$$\varphi(z) = \mathbb{E}\bigg[exp\bigg\{\frac{z}{\sqrt{n}}\sum_{k=1}^{nt} X_k\bigg\}\bigg] \tag{5}$$

Equation (5) we can rewritten as...

$$\varphi(z) = \mathbb{E}\bigg[\prod_{k=1}^{nt} exp\bigg\{\frac{z}{\sqrt{n}} X_k\bigg\}\bigg] \tag{6}$$

Because each increment in the random walk is independent we can rewrite Equation (6) as...

$$\varphi(z) = \prod_{k=1}^{nt} \mathbb{E}\bigg[exp\bigg\{\frac{z}{\sqrt{n}} X_k\bigg\}\bigg] \tag{7}$$

After further defining the expectation in Equation (7) above the moment generating function for the scaled symmetric random walk can be rewritten as...

$$\varphi(z) = \prod_{k=1}^{nt} \mathbb{E}\bigg[exp\bigg\{\frac{z}{\sqrt{n}} X_k\bigg\}\bigg] = \prod_{k=1}^{nt}\left(\frac{1}{2}e^{\frac{z}{\sqrt{n}}} + \frac{1}{2}e^{\frac{-z}{\sqrt{n}}}\right) = \left(\frac{1}{2}e^{\frac{z}{\sqrt{n}}} + \frac{1}{2}e^{\frac{-z}{\sqrt{n}}}\right)^{nt} \tag{8}$$

The Limit Of The Moment Generating Function As Step Size Goes To Zero

We want to know the limit of the moment generating function as defined by Equation (8) above as the length of the time step goes to zero. Since step size is $1 \div n$ this is equivalent to finding the limit of Equation (8) as n goes infinity. We start by taking the log of Equation (8) which is...

$$\ln\varphi(z) = nt \ln\left(\frac{1}{2}e^{\frac{z}{\sqrt{n}}} + \frac{1}{2}e^{\frac{-z}{\sqrt{n}}}\right) \tag{9}$$

We then make the following change of variables...

$$x = \frac{1}{\sqrt{n}} \;\; ...such\, that... \;\; n = \frac{1}{x^2} \tag{10}$$

Using the change of variables as defined by Equation (10) above Equation (9) can be rewritten as...

$$\ln\varphi(z) = t\,\frac{\ln(\frac{1}{2}e^{zx} + \frac{1}{2}e^{-zx})}{x^2} \tag{11}$$

As n goes to infinity the variable x, which is the square root of step size, goes to zero. The equation for the limit of Equation (11) as n goes to infinity and x goes to zero is...

$$\lim_{n\to\infty} \ln\varphi(z) = t \lim_{x\to 0} \frac{\ln(\frac{1}{2}e^{zx} + \frac{1}{2}e^{-zx})}{x^2} \tag{12}$$

Note that we have a problem. As x goes to zero the numerator and denominator of the right side of Equation (12) both go to zero such that you have $0 \div 0$, which is undefined. We will now make the following definitions...

$$f(x) = \ln\left(\frac{1}{2}e^{zx} + \frac{1}{2}e^{-zx}\right) \tag{13}$$

$$g(x) = x^2 \tag{14}$$

If as x goes to zero both $f(x)$ and $g(x)$ go to zero then L'Hopital's rule states that...

$$\lim_{n\to\infty} \ln\varphi(z) = t \lim_{x\to 0} \frac{\ln(\frac{1}{2}e^{zx} + \frac{1}{2}e^{-zx})}{x^2} = t \lim_{x\to 0}\frac{f(x)}{g(x)} = t \lim_{x\to 0}\frac{f'(x)}{g'(x)} \tag{15}$$

The derivative of our numerator as defined by Equation (13) is...

$$\begin{aligned}\frac{\delta f(x)}{\delta x} &= \frac{1}{\frac{1}{2}e^{zx}+\frac{1}{2}e^{-zx}} \times \frac{\delta}{\delta x}\left(\frac{1}{2}e^{zx}+\frac{1}{2}e^{-zx}\right)\\ &= \frac{1}{\frac{1}{2}e^{zx}+\frac{1}{2}e^{-zx}} \times \left(\frac{z}{2}e^{zx}-\frac{z}{2}e^{-zx}\right)\\ &= \frac{\frac{z}{2}e^{zx}-\frac{z}{2}e^{-zx}}{\frac{1}{2}e^{zx}+\frac{1}{2}e^{-zx}}\end{aligned} \tag{16}$$

The derivative of our denominator as defined by Equation (14) is...

$$\frac{\delta g(x)}{\delta x} = 2x \tag{17}$$

After invoking L'Hopital's rule Equation (12) can be written as...

$$\lim_{n\to\infty} \ln\varphi(z) = t\,\lim_{x\to 0}\left(\frac{1}{\frac{1}{2}e^{zx}+\frac{1}{2}e^{-zx}}\right)\left(\frac{\frac{z}{2}e^{zx}-\frac{z}{2}e^{-zx}}{2x}\right) \tag{18}$$

Noting that...

$$\lim_{x\to 0}\left(\frac{1}{\frac{1}{2}e^{zx}+\frac{1}{2}e^{-zx}}\right) = \left(\frac{1}{\frac{1}{2}+\frac{1}{2}}\right) = 1 \tag{19}$$

Equation (18) can be written as...

$$\lim_{n\to\infty} \ln\varphi(z) = t\,\lim_{x\to 0}\frac{\frac{z}{2}e^{zx}-\frac{z}{2}e^{-zx}}{2x} \tag{20}$$

Note that we still have a problem. As x goes to zero the numerator and denominator of the right side of Equation (20) both go to zero such that you still have $0 \div 0$. We will redefine $f(x)$ and $g(x)$ as follows...

$$f(x) = \frac{z}{2}e^{zx} - \frac{z}{2}e^{-zx} \tag{21}$$

$$g(x) = 2x \tag{22}$$

The derivative of our numerator as defined by Equation (21) is...

$$\frac{\delta f(x)}{\delta x} = \frac{z^2}{2}e^{zx} + \frac{z^2}{2}e^{-zx} \tag{23}$$

The derivative of our denominator as defined by Equation (22) is...

$$\frac{\delta g(x)}{\delta x} = 2 \tag{24}$$

After invoking L'Hopital's rule again Equation (20) can be written as...

$$\lim_{n\to\infty} \ln\varphi(z) = t\,\lim_{x\to 0}\frac{\frac{z^2}{2}e^{zx}+\frac{z^2}{2}e^{-zx}}{2} = \frac{1}{2}\,t\,z^2 \tag{25}$$

We now have a numerator and deonominator that are both greater than zero and can stop there.

The Moment Generating Function Of The Normal Distribution

The moment generating function for a normal distribution with mean m and variance v is...

$$\varphi(z) = e^{mz+\frac{1}{2}vz^2} \tag{26}$$

If we take the log of Equation (26) then log of the moment generating function for a normal distribution is...

$$\ln\varphi(z) = m\,z + \frac{1}{2}\,v\,z^2 \tag{27}$$

When the mean is zero and the variance is equal to the length of the time interval then Equation (27) becomes...

$$\ln\varphi(z) = \frac{1}{2}\,t\,z^2 \tag{28}$$

Conclusion

We set out to prove that as the step size in a scaled symmetric random walk goes to zero the distribution of the value of the random walk at time t converges to a normal distribution with mean zero and variance t. If this is the case then a scaled symmetric random walk converges to a Brownian motion as the length of each time step in the random walk gets smaller and smaller and the number of time steps in the random walk goes to infinity. We can see that Equation (25), which is the limiting distribution of a scaled symmetric random walk as step size goes to zero, equals Equation (28), with is the moment generating function for a normal distribution with mean zero and variance t, and accordingly we have proved our case.